Z-TRANSFORMATION

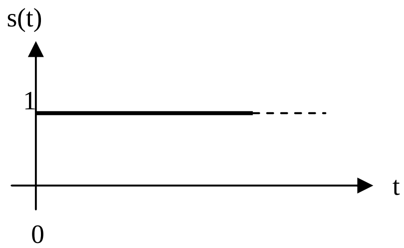
1. Z-transformation

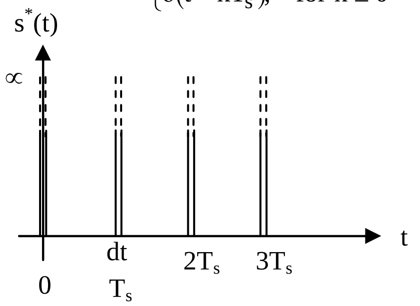
Z–transformation is Laplace transformation of some signal f(t) multiplied with discrete infinite impulses .

1.1. Defining Z-transformation

Continues step function s(t) and discontinues step function s*(t) are defined with folowing formulas

$$s(t) = \begin{cases} 0, & \text{for } t < 0 \\ 1, & \text{for } t \geq 0 \end{cases} \qquad s^*(kT_s) = \begin{cases} 0, & \text{for } k < 0 \\ \delta(t - kT_s), & \text{for } k \geq 0 \end{cases}$$





Therefore $s^*(t)$ can be written as folows:

$$s^*(t) = \sum_{k=0}^{\infty} \delta(t - kT). \tag{2.1}$$

Laplace transformation of some function ℓ(t) multiplied with $s^*(t)$ would be:

$$L\{\ell(t)s^*(t)\} = \int_{-\infty}^{\infty} \ell(t)s^*(t)e^{-st}dt. \tag{2.2}$$

After inserting (2.1) in (2.2) we get:

$$L\{\ell(t)s^*(t)\} = \int_{-\infty}^{\infty} \ell(t)\sum_{k=0}^{\infty}\delta(t - kT).e^{-st}dt.$$

By taking the sum out we get:

$$L\{\ell(t)s^*(t)\} = \sum_{k=0}^{\infty}\int_{-\infty}^{\infty} \ell(t)\delta(t - kT).e^{-st}dt$$

Since integral is acctualy a sum, as defined in(2.5), we get:

$$L\{\ell(t)s^*(t)\} = \sum_{k=0}^{\infty}\sum_{n=-\infty}^{\infty}\left[\ell(ndt)\delta(ndt - kT)e^{-sndt}dt\right]$$

Since δ(t) is different from 0 only if t = 0, as defined in (2.6), that means that all summands of second sum are 0 except when ntd=kT, so we get:

$$L\{\ell(t)s^*(t)\} = \sum_{k=0}^{\infty}\left[\ell(kT)\delta(kT - kT)e^{-skT}dt\right]$$

$$L\{\ell(t)s^*(t)\} = \sum_{k=0}^{\infty}\left[\ell(kT)\delta(0)e^{-skT}dt\right]$$

Since δ(0)=∝, as defined in (2.6), and dt=1/∝, these two factors desapire and we get:

$$L\{\ell(t)s^*(t)\} = \sum_{k=0}^{\infty}\left[\ell(kT)e^{-skT}\right] \tag{2.3}$$

By introducing symbol z, defined as:

$$z = e^{sT}$$

formula (2.3) can be simplified like this:

$$L\{\ell(t)s^*(t)\} = \sum_{k=0}^{\infty}\left[\ell(kT)z^{-k}\right] \tag{2.4}$$

Expression (2.4) can be shorted even more by calling this transformation Z-tranformation and introducing folowing substitution:

$$Z\{\ell(t)\} = L\{\ell(t)s^*(t)\}$$

Finaly, expression (2.4) can be written as:

$$Z\{\ell(t)\} = \sum_{k=0}^{\infty}\left[\ell(kT)z^{-k}\right]$$

This expression defines Z-transformation over function ℓ (t).

1.2. Formulas used in defining Z-transformation

$$\int_{t_s}^{t_e} f(t)dt = \sum_{k=0}^{\frac{t_s - t_e}{dt}-1} f(t + kdt)dt \tag{2.5}$$

$$\delta(t) = \begin{cases} \infty, & \text{for } t = 0 \\ 0, & \text{for } t \neq 0 \end{cases} \tag{2.6}$$

1.3. Laplace-trasform of discontinues step function

Wery important thing to remember is that Z-transform is never done on $s^*()$ because this has no practicly benefit. In this chapter we will make Laplace transformation of $s^*(t)$, and although the result could containz variable z, the result is not Z-transform of s*(t) becuase we didn't multiplie $s^*(t)$ with it self.

$$L\{s^*(t)\} = \int_{-\infty}^{\infty} s^*(t)e^{-st}dt$$

Substituting definition for $s^*(t)$ we get:

$$L\{s^*(t)\} = \int_{-\infty}^{\infty}\sum_{k=0}^{\infty}\delta(t-kT)e^{-st}dt = \sum_{k=0}^{\infty}\int_{-\infty}^{\infty}\delta(t-kT)e^{-st}dt = \sum_{k=0}^{\infty}e^{-skT} = 1 + e^{-sT} + e^{-2sT} + \ldots = \frac{1}{1-e^{-sT}}$$

Last step was made using geometric sequence rule.[Powell 159]
Although we could now substitute e^{-sT} with z we will not do it, so that the expression wouldn't look like Z-tranformationa, since this was not Z-transformation.